MATH 4060 MIDTERM EXAM (FALL 2016)

Name: ______________________________ Student ID: ______________________

Answer all questions. Write your answers on this question paper. No books, notes or calculators are allowed. Time allowed: 105 minutes.

1. Weierstrass's theorem states that a continuous function on $[-1, 1]$ can be uniformly approximated by polynomials there. Let $\overline{\mathbb{D}} = \{z \in \mathbb{C} \colon |z| \leq 1\}$ be the closed unit disc centered at the origin. Can every continuous function on $\overline{\mathbb{D}}$ be approximated uniformly on $\overline{\mathbb{D}}$ by polynomials in the complex variable z? Explain your answer. (10 points)

Solution.

- No.
- Polynomials in z are entire functions of z, and in particular holomorphic functions on $\mathbb{D}$.
- If a sequence of holomorphic functions on $\mathbb{D}$ converges uniformly on $\mathbb{D}$, then the limit function is holomorphic on $\mathbb{D}$.
- However, there are many functions that are continuous on $\overline{\mathbb{D}}$, but not holomorphic on $\mathbb{D}$.
- An example is $\overline{z}$. Such cannot be approximated uniformly on $\overline{\mathbb{D}}$ by polynomials in z.

2. (a) Let

$$f(x) = \frac{1}{1+x^2} \quad \text{for all } x \in \mathbb{R}.$$

Using contour integrals, show that its Fourier transform is

$$\widehat{f}(\xi) = \pi e^{-2\pi|\xi|},$$

where $\widehat{f}$ is defined by the formula $\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$ for all $\xi \in \mathbb{R}$. (16 points)

Solution.

- Given $\xi \in \mathbb{R}$, let $F(z) = \dfrac{1}{1+z^2} e^{-2\pi i z \xi}$.
- Suppose $\xi \leq 0$. For $R > 1$, let γ_R be the contour given by the straight line joining $-R$ to R, and C_R be the contour given by the semi-circle centered at 0 and of radius R, joining R to $-R$ through the upper half plane.
- The only singularity of F in the region enclosed by γ_R and C_R is at $z = i$.
- Writing $F(z) = \dfrac{e^{-2\pi i z \xi}/(z+i)}{z-i}$, the residue of F at $z = i$ is $\dfrac{e^{2\pi\xi}}{2i}$.
- Cauchy integral formula shows that $\dfrac{1}{2\pi i} \displaystyle\int_{\gamma_R + C_R} F(z) dz = \dfrac{e^{2\pi\xi}}{2i}$, hence
$$\int_{-R}^{R} f(x) e^{-2\pi i x \xi} dx = \pi e^{-2\pi|\xi|} - \int_{C_R} \frac{1}{1+z^2} e^{-2\pi i z \xi} dz.$$
- If $R > 1$, then for $z \in C_R$,
$$\left| \frac{1}{1+z^2} \right| \leq \frac{1}{R^2 - 1},$$
whereas
$$|e^{-2\pi i z \xi}| = e^{2\pi (\operatorname{Im} z)\xi} \leq 1 \quad \text{if } \xi \leq 0.$$
- Hence
$$\left| \int_{C_R} \frac{1}{1+z^2} e^{-2\pi i z \xi} dz \right| \leq \frac{\pi R}{R^2 - 1} \to 0$$
as $R \to +\infty$.
- This shows
$$\int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx = \pi e^{-2\pi|\xi|}$$
if $\xi \leq 0$. Similarly, if $\xi \geq 0$, instead of the upper semi-circle contour C_R, consider the contour given by the semi-circle centered at 0 and of radius R, joining R to $-R$ through the lower half plane. Then a similar calculation shows that the same identity holds; indeed, the residue one picks up at $z = -i$ is then $e^{-2\pi\xi}/(2i)$, and for z in this lower semi-circular contour, we have $|e^{-2\pi i z \xi}| = e^{2\pi (\operatorname{Im} z)\xi} \leq 1$ if $\xi \geq 0$. One then finishes the proof as before. (Alternatively, just observe that $f(x)$ is an even function, and thus its Fourier transform is also even; the result for $\xi \leq 0$ then implies also the result for $\xi \geq 0$.)

(b) (i) State (without proof) Poisson summation formula. You should state clearly a set of assumptions under which the conclusion of the theorem holds.
(ii) Using the version of Poisson summation formula you stated in part (i), evaluate the sum

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2}.$$

(14 points)

Solution.

- The Poisson summation formula says if $f(x)$ admits a holomorphic extension $f(z)$ to the horizontal strip $\{z \in \mathbb{C}\colon |\operatorname{Im} z| < a\}$ for some $a > 0$, and if the extension satisfies

$$|f(x+iy)| \leq \frac{A}{1+x^2} \quad \text{for all } x, y \in \mathbb{R} \text{ with } |y| < a,$$

then

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n),$$

where $\widehat{f}$ is the Fourier transform of f defined by $\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$.

- The Poisson formula above applies to $f(x) = \dfrac{1}{1+x^2}$, since it admits a holomorphic extension $f(z) = \dfrac{1}{1+z^2}$ to the strip $\{|\operatorname{Im} z| < 1/2\}$, and

$$|f(x+iy)| \leq \begin{cases} A & \text{if } |x| \leq 2, |y| < 1/2 \\ Ax^{-2} & \text{if } |x| > 2, |y| < 1/2 \end{cases}.$$

- Thus

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} = \sum_{n=-\infty}^{\infty} \pi e^{-2\pi|n|},$$

i.e.

$$1 + 2\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \pi\left(1 + 2\sum_{n=1}^{\infty} e^{-2\pi n}\right).$$

- We sum the geometric series on the right hand side:

$$\sum_{n=1}^{\infty} e^{-2\pi n} = \frac{e^{-2\pi}}{1-e^{-2\pi}}.$$

- Thus

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{1+n^2} &= \frac{\pi}{2}\left(1 + \frac{2e^{-2\pi}}{1-e^{-2\pi}}\right) - \frac{1}{2} \\
&= \frac{\pi}{2}\left(\frac{1+e^{-2\pi}}{1-e^{-2\pi}}\right) - \frac{1}{2} \\
&= \frac{\pi\coth(\pi) - 1}{2} \quad \text{or} \quad \frac{\pi - 1}{2} + \frac{\pi e^{-2\pi}}{1-e^{-2\pi}}.
\end{aligned}$$

3. Suppose $k \in \mathbb{N}$. Let E_k be the canonical factor, defined by

$$E_k(z) = (1-z)\exp\left(\sum_{j=1}^{k} \frac{z^j}{j}\right) \quad \text{for } z \in \mathbb{C}.$$

(a) Show that

$$|E_k(z)| \geq e^{-2|z|^{k+1}} \quad \text{for all } |z| \leq \frac{1}{2}.$$

(12 points)

Solution.

- Suppose $z \in \mathbb{C}$ and $|z| \leq 1/2$. Then
$$1 - z = \exp(\operatorname{Log}(1-z))$$
where Log is the principal branch of the logarithm.
- Hence
$$\operatorname{Log}(1-z) = -\sum_{j=1}^{\infty} \frac{z^j}{j}.$$
- This gives
$$E_k(z) = \exp\left(-\sum_{j=k+1}^{\infty} \frac{z^j}{j}\right).$$
- But
$$\begin{aligned}\operatorname{Re}\left(\sum_{j=k+1}^{\infty} \frac{z^j}{j}\right) &\leq \left|\sum_{j=k+1}^{\infty} \frac{z^j}{j}\right| \\ &\leq \sum_{j=k+1}^{\infty} \frac{|z|^j}{j} \\ &\leq \sum_{j=k+1}^{\infty} |z|^j \\ &\leq |z|^{k+1} \sum_{j=0}^{\infty} \frac{1}{2^j} \\ &= 2|z|^{k+1}\end{aligned}$$
- Thus
$$|E_k(z)| = \exp\left(-\operatorname{Re}\left(\sum_{j=k+1}^{\infty} \frac{z^j}{j}\right)\right) \geq e^{-2|z|^{k+1}}.$$

(b) Show that if $z \in \mathbb{C}$, and $\{a_n\}_{n=1}^{\infty}$ is a sequence of complex numbers satisfying both conditions below:

- $|a_n| \geq 2|z|$ for all $n \in \mathbb{N}$,
- $\sigma := \displaystyle\sum_{n=1}^{\infty} \frac{1}{|a_n|^k} < \infty$,

then

$$\left| \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right) \right| \geq e^{-\sigma |z|^k}.$$

(You do not need to prove the convergence of the infinite product on the left hand side.) (8 points)

Solution.

- Suppose $|a_n| \geq 2|z|$ for all $n \in \mathbb{N}$. Then $|z/a_n| \leq 1/2$ for all n.
- Thus for any positive integer N, we have
$$\left| \prod_{n=1}^{N} E_k\left(\frac{z}{a_n}\right) \right| \geq \prod_{n=1}^{N} \exp\left(-2\left|\frac{z}{a_n}\right|^{k+1}\right) = \exp\left(-2\sum_{n=1}^{N}\left|\frac{z}{a_n}\right|^{k+1}\right)$$
- But
$$\sum_{n=1}^{N} \left|\frac{z}{a_n}\right|^{k+1} \leq \frac{1}{2}\sum_{n=1}^{N}\left|\frac{z}{a_n}\right|^{k} = \frac{\sigma |z|^k}{2}.$$
- Thus
$$\left| \prod_{n=1}^{N} E_k\left(\frac{z}{a_n}\right) \right| \geq e^{-\sigma |z|^k}.$$
This is true for any positive integer N. Letting $N \to +\infty$ then yields the result.

4. For each of the following statements, determine whether it is true or false. If it is true, give a proof; if it is false, show that it is false.

(a) The infinite product
$$\prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)$$
converges for all $z \in \mathbb{C}$, and defines an entire function that vanishes to order 1 at all the positive integers. (10 points)

Solution.

- False.
- The infinite product does not converge at $z=-1$.
- If $z=-1$, then for any positive integer N, we have
$$\prod_{n=1}^{N}\left(1-\frac{z}{n}\right)=\prod_{n=1}^{N}\left(1+\frac{1}{n}\right)=\prod_{n=1}^{N}\frac{n+1}{n}=N+1,$$
which diverges as $N \to +\infty$.
- Indeed, the infinite product fails to converge for any $z \notin \mathbb{N} \cup \{0\}$. This follows from the following fact (see Stein and Shakarchi, Complex Analysis, Chapter 5, Exercise 7(a)):

  Suppose (i) $a_n \neq -1$ for all $n \in \mathbb{N}$, (ii) $\sum_{n=1}^{\infty}|a_n|^2$ converges, and (iii) $\sum_{n=1}^{\infty} a_n$ diverges. Then $\prod_{n=1}^{\infty}(1+a_n)$ diverges.

  It suffices to apply this fact to $a_n := z/n$.

(b) If P is a monic polynomial (i.e. a polynomial of the form $z^n + c_{n-1}z^{n-1} + \cdots + c_0$ for some $n \in \mathbb{N} \cup \{0\}$ and some coefficients $c_0, c_1, \dots, c_{n-1} \in \mathbb{C}$) and $P(z) \neq 0$ whenever $|z| \geq 1$, then

$$\int_0^{2\pi} \log |P(e^{it})| dt = 0.$$

(14 points)

Solution.

- True.
- Since P is monic and all zeroes of P are inside the open unit disc $\mathbb{D}$, we may write
$$P(z) = (z - a_1)(z - a_2) \dots (z - a_n)$$
for some $a_1, \dots, a_n \in \mathbb{D}$.
- Now
$$\int_0^{2\pi} \log |P(e^{it})| dt = \sum_{k=1}^{n} \int_0^{2\pi} \log |e^{it} - a_k| dt.$$
- So it suffices to show that
$$\int_0^{2\pi} \log |e^{it} - a| dt = 0$$
for all $a \in \mathbb{D}$.
- This is clear if $a = 0$.
- If $a \in \mathbb{D}$ and $a \neq 0$, then we apply Jensen's formula to $p(z) := z - a$. Then we get
$$\int_0^{2\pi} \log |e^{it} - a| dt = \int_0^{2\pi} \log |p(e^{it})| dt = 2\pi(\log |p(0)| - \log |a|) = 0$$
as well, as desired.
- Alternatively, consider the function $Q(z) := z^n P(1/z)$, defined for $z \neq 0$. Then since P is monic, we have
$$\lim_{z \to 0} Q(z) = 1,$$
so we may extend Q to an entire function. Now Q has no zeroes on $\overline{\mathbb{D}}$. Thus $\log |Q(z)|$ is harmonic on a neighborhood of $\overline{\mathbb{D}}$, and the mean-value property for harmonic functions shows that
$$\int_0^{2\pi} \log |Q(e^{it})| dt = 2\pi \log |Q(0)| = 0.$$
(We could also invoke Jensen's formula here to obtain the same conclusion.) Since $|Q(e^{it})| = |P(e^{-it})|$, and $\int_0^{2\pi} \log |P(e^{it})| dt = \int_0^{2\pi} \log |P(e^{-it})| dt$ by a change of variable $t \mapsto 2\pi - t$, the desired result follows.

5. Suppose f is entire, not identically zero, and each zero of f occurs with an even multiplicity. Show that there exists an entire function g such that $g^2 = f$. (16 points)

Solution.

- Let $\Lambda = \{w \in \mathbb{C}\colon f(w) = 0\}$ be the zero set of f.
- Since f is not identically zero, Λ has no accumulation points.
- If $w \in \Lambda$, then f vanishes to order $2m_w$ at w for some positive integer m_w.
- Weierstrass's theorem guarantees the existence of some entire function h, such that the following holds:
  - if $w \in \Lambda$, then h vanishes to order m_w at w, while
  - if $w \in \mathbb{C} \setminus \Lambda$, then h does not vanish at w.
- The function f/h^2 is then entire, and has no zeroes at all.
- Since $\mathbb{C}$ is simply connected, one can write f/h^2 as e^H for some entire function H.
- Thus $f = e^H h^2$, and it follows that $f = g^2$ if $g := e^{H/2}h$. Such g is obviously entire.

End of paper